Int. J. Solids Structures Vol. 29, No. 3, pp. 271–277, 1992
Printed in Great Britain.

0020–7683/92 $5.00+ .00

AN INVESTIGATION INTO THE STRESS-FIELD SINGULARITY AT THE MOUTH OF A SURFACE-BREAKING CRACK

A. N. GALYBIN
Department of Mathematics, Moscow Mining Institute, Moscow, U.S.S.R.

A. V. DYSKIN
Department of Mining Geometry, Moscow Mining Institute, Moscow, U.S.S.R.

and

A. M. KORSUNSKY
Department of Engineering Science, University of Oxford, Oxford OX1 3PJ, England

(*Received* 2 *March* 1991; *in revised form* 2 *June* 1991)

Abstract—Two-dimensional elastic stress field behaviour near the mouth of a surface crack in an isotropic half-plane is examined. It is shown that a dislocation density simulating the crack opening and consequently the stress field has a singularity at the point in question. The type of singularity is determined by the distribution of the load applied as well as by the angle between the crack contour and the surface. Two known solutions for stress intensity factors of a uniformly loaded crack are compared, the first taking the singularity into account and the second neglecting it. It is found that the solutions differ significantly when the crack is inclined at an acute angle.

1. INTRODUCTION

The contact of two bodies under severe loading often produces surface-breaking cracks which influence the character of the deformation and may eventually lead to failure. Phenomena of this kind occur in many situations such as cracking of gear teeth or machine tools, crack growth in foundations of dikes or in civil engineering.

Usually modelling a surface crack numerically the assumption is made that the stress field behaviour in the vicinity of the crack mouth is regular [see, for example, Savruk (1981) or Nowell and Hills (1987) and references].

Certainly the following purely geometric considerations led to the conclusion that there should be no stress field singularity. Indeed, the crack mouth can be presented as a combination of two wedges (BAD and CA'D in Fig. 1). Each of them has an angle less than π and hence no singularities in their eigensolutions arise (Ufland, 1963). However the situation is also dependent on the boundary conditions. If, for instance, the crack is subjected to a uniform loading (Fig. 1) or the half-plane boundary is loaded by a stamp so that the crack is growing from its edge (which corresponds to many practically important

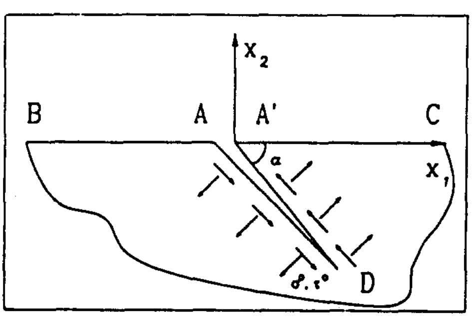

Fig. 1. A surface-breaking crack inclined at an angle α and loaded by normal and shear tractions along its shores.

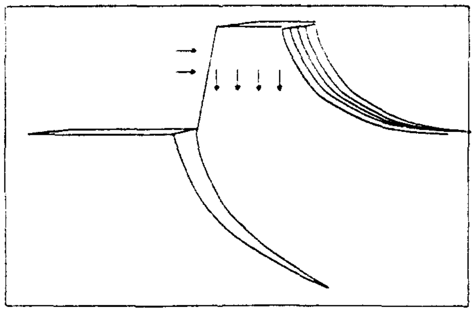

Fig. 2. An example of a crack propagating from the surface subjected to local loading.

cases, see Fig. 2), then one can expect that the discontinuous boundary conditions will produce singularities in stress and deformation fields. This consideration calls for a more thorough examination of the problem.

The stress–strain field in a wedge under various boundary conditions has been considered by the power extension method by Williams (1952). However, power series fail to reflect every possible type of singularity, which, as will be shown, may be logarithmic. Direct application of the Mellin transform for the analysis of two edge-bonded elastic wedges (Bogy, 1971) allows the right type of singularities to be obtained.

We tried to focus our attention on the case of finite surface crack. The general problem for a wedge with a crack grown from its edge (Khrapkov, 1971) was solved in quadratures. The solution implicitly contains the right behaviour of stress field at the vertex. However, the applicability of the analytical solutions is limited and in order to apply numerical approaches properly one has to know the explicit form of the singularity. The determination of this form and classification of possible cases are the aims of the present paper.

2. FORMULATION

Consider a half-plane with a linear surface crack inclined to the boundary at an angle α (Fig. 1). Let us suppose the crack length to be unity. The crack surfaces are loaded by normal and shear stresses $\sigma^0(\xi)$, $\tau^0(\xi)$, where ξ is the coordinate along the crack axis measured from the crack mouth (hence $0 \leqslant \xi \leqslant 1$ corresponds to the crack).

The problem can be reduced to an integral equation in the complex plane in the following manner. The crack is replaced by a continuous distribution of edge dislocations of unknown density. The expressions for stresses in a half-plane due to a single dislocation are well known. Summing up the contributions of each dislocation to the stress values for every point on the line corresponding to the crack contour and equating the result to the applied stresses $(\sigma^0(\xi), \tau^0(\xi))$ the integral equation for the unknown dislocation density can be obtained. This equation [see, for example, Savruk (1981)] in the notation adopted here has the form:

$$\beta \int_0^1 \left\{ g'(\xi) \left[\frac{1}{\xi - \zeta} + \frac{1}{\eta - e^{2i\alpha}} + \eta^2 + \xi\eta(1 - \eta\, e^{-2i\alpha} + e^{-4i\alpha}) + \frac{\xi^2 (e^{-2i\alpha} - e^{-4i\alpha} + e^{-6i\alpha})}{(\eta - \xi\, e^{-2i\alpha})^3} \right] + \xi \overline{g'(\xi)} \left[\frac{1 - e^{-2i\alpha}}{(\eta - \xi\, e^{-2i\alpha})} + \frac{e^{2i\alpha}(1 - e^{2i\alpha})}{(\eta - \xi\, e^{2i\alpha})^2} \right] \right\} d\xi = \pi[\sigma^0(\eta) + i\tau^0(\eta)]. \quad (1)$$

Here the complex dislocation density is denoted by $g'(\xi) = u'_1 + iu'_2$, i being $\sqrt{-1}$ and (u_1, u_2) the components of displacement discontinuity vector in the Cartesian coordinate system (x_1, x_2), shown in Fig. 1, the prime denoting differentiation by ξ, the overbar denoting complex conjugation and $\beta = G/4\pi(1-\nu)$ for plane strain (G is the shear modulus,

ν—Poisson's ratio). For the purposes of further consideration (1) can be rewritten introducing kernel functions K_1 and K_2:

$$\frac{\beta}{\pi}\int_0 \{g'(\xi)K_1(\xi,\eta,\alpha)+\overline{g'(\xi)}K_2(\xi,\eta,\alpha)\}\,d\xi = [\sigma^0(\eta)+i\tau^0(\eta)] \tag{2}$$

where

$$\begin{aligned} K_1(\xi,\eta,\alpha) &= \frac{1}{\xi-\eta}+B(\xi,\eta,-\alpha)+[\eta^2+\xi\eta(1-4\,e^{2i\alpha}+e^{-4i\alpha}) \\ &\quad +\xi^2(e^{-2i\alpha}-e^{-4i\alpha}+e^{-6i\alpha})]B^3(\xi,\eta,\alpha), \\ K_2(\xi,\eta,\alpha) &= \xi[(1-e^{-2i\alpha})B^2(\xi,\eta,\alpha)+e^{2i\alpha}(1-e^{2i\alpha})B^2(\xi,\eta,-\alpha)], \\ B(\xi,\eta,\alpha) &= \frac{1}{\eta-\xi\,e^{-2i\alpha}}. \end{aligned}$$

3. ANALYSIS OF THE SINGULARITY

It is convenient now to present eqn (2) as a system of two real integral equations:

$$\frac{\beta}{\pi}\int_0^1 K(\xi,\eta)u'(\xi)\,d\xi = \sigma_0(\eta) \tag{3}$$

where

$$u' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix},\quad \sigma_0 = \begin{pmatrix} \sigma^0 \\ \tau^0 \end{pmatrix},\quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \tag{4}$$

$$\begin{aligned} K_{11} &= \mathrm{Re}\,[K_1(\xi,\eta,\alpha)+K_2(\xi,\eta,\alpha)], \\ K_{12} &= -\mathrm{Im}\,[K_1(\xi,\eta,\alpha)-K_2(\xi,\eta,\alpha)], \\ K_{21} &= \mathrm{Im}\,[K_1(\xi,\eta,\alpha)+K_2(\xi,\eta,\alpha)], \\ K_{22} &= \mathrm{Re}\,[K_1(\xi,\eta,\alpha)-K_2(\xi,\eta,\alpha)]. \end{aligned} \tag{5}$$

Analytical investigation of the integral equations for problems of this type may often be accomplished by reducing the system to be solved to the conjugation problem by application of the Mellin transform:

$$f(\zeta) \Rightarrow F(p) = \int_0^\infty f(\zeta)\zeta^p\,d\zeta. \tag{6}$$

In the case discussed the unknown vector u' which is initially introduced for $0 \leqslant \xi \leqslant 1$ must be additionally assumed to be zero over the semi-infinite interval $\xi \geqslant 1$. Thus the Mellin transform for the unknown vector u' becomes:

$$\phi^+(p) = \int_0^1 u'(\zeta)\zeta^p\,d\zeta. \tag{7}$$

It is also convenient to define the functions $\sigma^+(p)$ and $\sigma^-(p)$ as the transforms of the stress distributions in the appropriate intervals. We introduce:

$$\sigma^{+}(p) = \int_0^1 \sigma_0(\zeta)\zeta^p \, d\zeta \tag{8}$$

where $\sigma_0(\zeta)$ is the given vector of tractions applied on the crack line, and

$$\sigma^{-}(p) = \int_1^{\infty} \sigma(\zeta)\zeta^p \, d\zeta \tag{9}$$

is an unknown function appearing due to the extension of eqn (8) to a semi-infinite interval and having the sense of the transform of the unknown stress vector σ acting inside the body over the continuation of crack line *ad infinitum*.

Making use of the variable transformation in the form $\xi = \zeta \cdot \eta$ the system (3) may be rewritten as:

$$\frac{\beta}{\pi}\int_0^1 K(1, 1/\zeta)u'(\zeta \cdot \eta)\frac{d\zeta}{\zeta} = \sigma_0(\eta). \tag{10}$$

Thus it can be seen that the kernel is of the Mellin convolution type.

Applying the Mellin transform to (10) and in view of the definitions being made one obtains:

$$\beta Q(p)\phi^{+}(p) = \sigma^{+}(p) + \sigma^{-}(p). \tag{11}$$

Here the matrix $Q(p)$ is the Mellin transform for $K(1, \xi)$. A more convenient way of presenting (11) is:

$$\phi^{+}(p) = \frac{1}{\beta}G(p)[\sigma^{+}(p) + \sigma^{-}(p)] \tag{12}$$

where $G(p) = Q^{-1}(p)$ and has the form [see Khrapkov (1971)]:

$$G(p) = \begin{pmatrix} G_{11}(p) & G_{12}(p) \\ G_{21}(p) & G_{22}(p) \end{pmatrix} \tag{13}$$

$$\begin{aligned} G_{11}(p) &= \tfrac{1}{4}\{D(p,\alpha)[\sin 2p\alpha + p\sin 2\alpha] + D(p,\pi-\alpha)[\sin 2p(\pi-\alpha) + p\sin 2(\pi-\alpha)]\} \\ G_{12}(p) &= p(p-1)\sin^2\alpha[D(p,\alpha) - D(p,\pi-\alpha)] \\ G_{21}(p) &= -p(p+1)\sin^2\alpha[D(p,\alpha) - D(p,\pi-\alpha)] \\ G_{22}(p) &= \tfrac{1}{4}\{D(p,\alpha)[\sin 2p\alpha - p\sin 2\alpha] + D(p,\pi-\alpha)[\sin 2p(\pi-\alpha) - p\sin 2(\pi-\alpha)]\} \end{aligned} \tag{14}$$

$$D(p,\alpha) = (p^2\sin^2\alpha - \sin^2 p\alpha)^{-1}. \tag{15}$$

The functions denoted by the superscript "+" are regular for $\mathrm{Re}\, p \geqslant 0$. This follows from a physically obvious fact that the tractions σ_0 and dislocation density u' are integrable over the interval (0, 1).

The function σ^{-} is regular for $\mathrm{Re}\, p \leqslant 0$, since the stresses σ must be integrable over the interval $(1, \infty)$ according to the equilibrium conditions.

In order to make further conclusions the behaviour of the matrix G in the complex plane must be discussed. The matrix has a pole of first order in $p = 0$. However, $\sigma^{+}(0) + \sigma^{-}(0) = 0$ since the half-plane remains in equilibrium. Therefore, eqn (10) is satisfied for $\mathrm{Re}\, p = 0$ (along the imaginary axis).

Equation (12) was obtained (Khrapkov, 1971) by an alternative means (applying the Mellin transform to the general elasticity equations) and was solved analytically using a

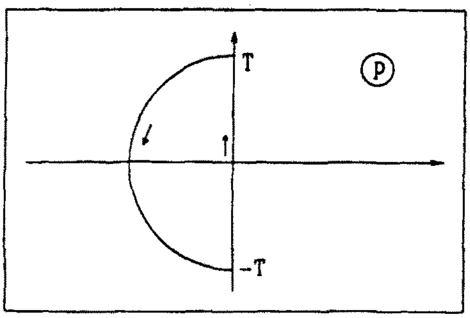

Fig. 3. The contour in the complex p-plane chosen for evaluating the inverse Mellin transforms using the Jordan's lemma.

factorization technique. For the analysis of a singularity at the crack mouth ($\zeta = 0$) however it is sufficient to apply the reverse Mellin transform to (12):

$$u'(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \phi^+(p)\zeta^{-p-1}\,dp$$

$$= \frac{1}{2\pi\beta i}\int_{-\infty}^{+\infty} G(p)[\sigma^+(p)+\sigma^-(p)]\zeta^{-p-1}\,dp. \tag{16}$$

It is now convenient to express the integral as a sum of residues, choosing the contour in the complex plane p as shown in Fig. 3. Assuming $T \to \infty$ and applying the Jordan's lemma one can obtain:

$$u'(\zeta) = \frac{1}{\beta}\sum_{p_i} \mathrm{Res}_{p_i}\{G(p)[\sigma^+(p)+\sigma^-(p)]\zeta^{-p-1}\}. \tag{17}$$

The terms in (17) which correspond to such poles p_i that $\mathrm{Re}\,p_i < -1$ tend to zero for $\zeta \to 0$ and therefore make no contribution to the singularities. Hence one can confine the analysis to the poles with $-1 \leqslant \mathrm{Re}\,p_i \leqslant 0$. Function $\sigma^-(p)$ is regular for $\mathrm{Re}\,p < 0$ and therefore the singularities of u' may be connected either with poles of $G(p)$ or with poles of $G(p)\sigma^+(p)$. The former may be considered as the term corresponding to the geometry induced singularities (i.e. they arise in the eigensolution of the problem), while the latter term reflects the influence of both the geometry and the boundary conditions (i.e. the traction variation along the crack).

In general the components $G_{11}(p)$ and $G_{12}(p)$ have the only first order pole for $p = -1$ and $G_{21}(p)$, $G_{22}(p)$ display regular behaviour in the whole region specified. In the special case of $\alpha = \pi/2$ the components $G_{12} = G_{21} = 0$ and diagonal components G_{11} and G_{22} are regular.

Thus one can conclude that, as anticipated, there are no pure geometry-induced singularities (as found by the eigensolution) for $u'(\zeta)$ at $\zeta = 0$.

The poles of $\sigma^+(p)$ are determined by the traction profile in the vicinity of crack mouth. Several important cases can be emphasized:

(1) If the components of $\sigma_0(\zeta)$ tend to zero as any power of ζ, then $\sigma^+(p)$ is regular for $-1 \leqslant \mathrm{Re}\,p \leqslant 0$. In this case no singularities arise in dislocation density.

(2) If the components of $\sigma_0(\zeta)$ are limited but non-zero when $\zeta \to 0$ then $\sigma^+(p)$ has a pole of first order for $p = -1$. Hence in general $G(p)\sigma^+(p)$ has a second-order pole in this point. It leads to the logarithmic singularity for the dislocation density component u'_1 (but not for u'_2) at $\zeta = 0$, i.e. in the vicinity of this point $u'_1 \simeq a_0 + a_1 \ln\zeta$, $u'_2 \simeq b_0$.

A practically important case arising in problems with uniform tensile stress $\sigma_{xx} = \sigma^\infty$ at infinity should be noted. The problems of this kind may be reduced to the type discussed by decomposing the stress field into a sum of uniform stresses $\sigma_{xx} = \sigma^\infty$ and the field arising from uniform tractions $\sigma_n(\zeta) = -(\sigma^\infty/2)(1-\cos 2\alpha)$, $\tau(\zeta) = (\sigma^\infty/2)\sin 2\alpha$ applied along the crack surface. In this case the kernel in (16) is in effect represented by the product of matrix G by a vector column of tractions

$$\sigma^+(p) = \begin{pmatrix}\sigma_n(p)\\ \tau(p)\end{pmatrix} = -\frac{\sigma^\infty}{2(p+1)}\begin{pmatrix}1-\cos 2\alpha\\ -\sin 2\alpha\end{pmatrix}. \tag{18}$$

As can be derived by virtue of expressions (14) for the components of matrix G and (18) for the traction vector, the terms, corresponding to the first order pole of $G(p)$, cancel out in the product $G(p)\cdot\sigma^+(p)$. Thus, as could be expected, in this case the stress field (as well as the dislocation density) behave regularly.

(3) If the components of $\sigma_0(\zeta)$ have a logarithmic singularity near $\zeta = 0$ (this corresponds to a practically important case of a crack growing from an edge of half-plane boundary segment loaded by built-up forces produced, for example, by a stamp), then $\sigma^+(p)$ has a second-order pole for $p = -1$, which gives rise to a singularity in dislocation density: u_1' generally behaves like $a_0+a_1\ln\zeta+a_2\ln^2\zeta$ and $u_2' \simeq b_0+b_1\ln\zeta$. For $\alpha = \pi/2$: $u_1' \simeq a_0+a_1\ln\zeta$, $u_2' \simeq b_0+b_1\ln\zeta$.

For stronger singularities in the traction distribution $\sigma_0(\zeta)$ in the vicinity of the surface one can expect $\sigma^+(p)$ to have additional poles. For instance, if $\sigma_0(\zeta) \simeq \zeta^{-1/2}$ then $\sigma^+(p)$ has a pole at $p = -1/2$, and the dislocation density appropriately behaves like $\zeta^{-1/2}$.

In a general case of traction profile the problem can be similarly analyzed in the manner discussed above.

It must be noted that the stress field in the vicinity of the crack mouth generally has the same order singularities as the dislocation density. This can be readily derived from the presentation of stresses and dislocation densities as functions of the Kolossov–Muskhelishvili potentials (Muskhelishvili, 1953).

4. DISCUSSION

The analysis presented shows that in the general situation dislocation density as well as the stress field components display singularities at the crack mouth, the order of these singularities depending on the profile of the stresses along the crack or at infinity. Another interesting result is that even if the tractions at the crack mouth are nil, the dislocation density may differ from zero at this point, which sometimes calls for amendments to the computational procedures (Savruk, 1981).

However, it seems rational to analyze the influence of the singularity on the stress state and stress intensity factors depending on the entire traction profile and the crack angle to the surface. This analysis can be done for a common case of uniform traction by comparing two solutions for stress intensity factors, one of which (Savruk, 1981) does not take the singularity into account and the other (Khrapkov, 1971) implicitly using it. In order to facilitate this comparison it is necessary to integrate Khrapkov's solution (which is performed for a point force on the crack contour) over the crack length. The results can be presented in a form:

$$\begin{pmatrix}K_1\\ K_2\end{pmatrix} = \sqrt{\pi}A\begin{pmatrix}\sigma^0\\ \tau^0\end{pmatrix}. \tag{19}$$

Here A is a matrix and its values for different angles α are presented in Table 1.

It is seen that the difference between these two approaches is marked when the crack is at a very shallow angle to the surface, becoming important for $\alpha < \pi/4$.

Table 1.

α	Matrix A			
	Solution (1) No singularity		Solution (2) With singularity	
$\pi/2$	1.121	0	1.121	0
	0	1.121	0	1.121
$\pi/4$	1.596	−0.186	1.615	−0.218
	−0.508	1.237	−0.501	1.243
$\pi/12$	5.050	−0.421	3.192	−0.248
	−3.161	1.754	−5.089	1.831

5. CONCLUSION

The above analysis showed, that both the dislocation density and the stress field at the mouth of a surface crack display singular behaviour of an order depending on the geometry and tractions applied at infinity or along the crack contour. The presence of this singularity significantly affects the stress field around the crack. However, the influence on the SIF for the case of uniform loading can be neglected for angles greater than $\pi/4$. For more complicated cases the order of this singularity must be assessed prior to applying some numerical approach, and appropriate weight functions or elements must be chosen.

REFERENCES

Bogy, D. B. (1971). Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *J. Appl. Mech.* pp. 377–386.

Khrapkov, A. A. (1971). Some cases of elastic equilibrium of an infinite wedge with an unsymmetric notch at the vertex subjected to concentrated forces. *PMM J. Appl. Math. Mech.* **35**(4), 625–637.

Muskhelishvili, N. I. (1953). *Some Basic Problems of Mathematical Theory of Elasticity*. Noordhoff, Groningen.

Nowell, D. and Hills, D. A. (1987). Open cracks at or near free edges. *J. Strain Analysis* **22**(3), 171–185.

Savruk, M. P. (1981). *Two-dimensional Elastic Problems for Bodies with Cracks*. Naukova Dumka, Kiev (in Russian).

Ufland, Ya. S. (1963). *Integral Transforms in Elastic Problems*. Academy of Sciences, Moscow (in Russian).

Williams, M. L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.* **19**, 526–530.